Kuwait University | Math 102 | 11 December 2010
Department of Mathematics | Second Exam | 9–10.30am

**Answer all questions. Calculators and mobile phones are NOT allowed.**

1. (3 pts. each) Find the following limits, if they exist.

(a) $\displaystyle\lim_{x\to 0} \frac{x-\sin^{-1}x}{1-\cosh x}$

(b) $\displaystyle\lim_{x\to 0}(1-\cos x)^{1-\sec x}$

2. (4 pts.) Evaluate $\displaystyle\int_0^{\pi/2} \sin x \cos x \cos(4x)\,dx$.

3. (3 pts. each) Evaluate the following integrals:

(a) $\displaystyle\int x^2 \sinh x\,dx$

(b) $\displaystyle\int_0^{2-\sqrt{2}} \frac{x^2-4x+5}{(2-x)\sqrt{x^2-4x+3}}\,dx$

(c) $\displaystyle\int \frac{4x^2+2x+4}{x(x^2+4)}\,dx$

(d) $\displaystyle\int \frac{dx}{(1-x^{1/3})x^{1/2}}$

(e) $\displaystyle\int \frac{\csc x}{(1+\cos x)^2}\,dx$

Solutions

1. (a) $\lim_{x\to 0} \frac{x - \sin^{-1} x}{1 - \cosh x} = \lim_{x\to 0} \frac{1 - \frac{1}{\sqrt{1-x^2}}}{- \sinh x} = \lim_{x\to 0} \frac{-x/(1-x^2)^{3/2}}{- \cosh x} = 0.$

   (b) Put $y = (1 - \cos x)^{1-\sec x}$. Then $\ln y = (1 - \sec x) \ln(1 - \cos x)$
   $= \lim \frac{\ln(1-\cos x)}{(1-\sec x)^{-1}} = \lim \frac{\sin x/(1-\cos x)}{(1-\sec x)^{-2} \sec x \tan x}$
   $= \lim \frac{(\cos x - 1)^2}{1 - \cos x} = 0$, and $\lim y = 1$.

2. $\sin x \cos x \cos 4x = \frac{1}{2} \sin 2x \cos 4x = \frac{1}{4}(\sin 6x - \sin 2x)$, so our integral is $-\frac{1}{4}(\frac{1}{6} \cos 6x - \frac{1}{2} \cos 2x)|_0^{\pi/2} = -1/6$.

3. (a) By parts: $\int = \int x^2 d(\cosh x) = x^2 \cosh x - 2 \int x \cosh x dx$
   $= x^2 \cosh x - 2(\int x d(\sinh x)) = x^2 \cosh x - 2(x \sinh x - \cosh x) =$
   $x^2 \cosh x - 2x \sinh x + 2 \cosh x$.

   (b) Put $2 - x = \sec y$. Then $\int_0^{2-\sqrt{2}} dx = -\int \frac{(1+\sec^2 y) \tan y \sec y \, dy}{\sec y \tan y}$
   $= -\int (1 + \sec^2 y) dy = y + \tan y|_{\pi/4}^{\pi/3} = \frac{\pi}{12} + \sqrt{3} - 1$.

   (c) $\frac{4x^2+2x+4}{x(x^2+4)} = \frac{a}{x} + \frac{bx+c}{x^2+4}$, so $4x^2 + 2x + 4 = a(x^2 + 4) + x(bx + c)$ which gives $a = 1, b = 3, c = 2$. Thus $\int = \int (\frac{1}{x} + \frac{3}{2} \frac{2x}{x^2+4} + \frac{2}{x^2+4}) dx =$
   $\ln x + \frac{3}{2} \ln(x^2 + 4) + \tan^{-1} \frac{x}{2}$.

   (d) Put $u^6 = x$. Then
   $6u^5 du = dx, \int = \int \frac{6u^5 du}{u^3(1-u^2)} = \int \frac{6u^2}{1-u^2} = \int (3\frac{1}{1-u} + 3\frac{1}{1+u} - 6) du$
   $= 3 \ln |1 + \sqrt[3]{x}| - 3 \ln |1 - \sqrt[3]{x}| - 6\sqrt[6]{x}$.

   (e) Put $x = \tan \frac{t}{2}$. Then $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2dt}{1+t^2}$, so
   $$\int = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}(1 + \frac{1-t^2}{1+t^2})^2} = \int \frac{1 + 2t^2 + t^4}{4t} dt = \frac{1}{4} \ln t + \frac{1}{4} t^2 + \frac{1}{16} t^4.$$